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STRESS AND INDUCTION FIELD OF A SPHEROIDAL INCLUSION OR A PENNY-SHAPED CRACK IN A TRANSVERSELY ISOTROPIC PIEZO-ELECTRIC MATERIAL

L. KOGAN, C.-Y. HUI†
Theoretical and Applied Mechanics, Cornell University, Ithaca, NY 14853, U.S.A.

and

V. MOLKOV
Mechanics and Mathematics, Moscow State University, Moscow, 119899, Russia

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Abstract—Exact closed-form solutions are obtained for the stress and induction field of a spheroidal piezo-electric inclusion in an infinite piezo-electric matrix subjected to spatially homogeneous mechanical and electrical loadings far away from the inclusion. Three types of loading are considered: axisymmetric, in-plane and out-of-plane shear. A limiting case of this solution allows us to determine the stress and induction field of a penny-shaped crack in a piezo-electric material. Closed-form expressions for the stress and induction intensity factors of a penny-shaped crack are obtained. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

The stress field caused by an ellipsoidal inclusion in a linearly elastic infinite body is of great interest in the mechanics of composites. For example, the solution of the above problem can be used to estimate the effective properties of particulate composites, e.g. Willis (1977), Christensen (1979). Eshelby (1957) first showed that, when both the inclusion and the infinite body are linearly elastic and isotropic, the stress and strain fields inside the inclusion are uniform when the inclusion undergoes a uniform transformation strain. Eshelby's formulation can be generalized to anisotropic inclusions in anisotropic materials [see Sadowsky (1947), Lurie (1952), Bose (1965), Mura (1976), etc.]. These formulations are exact, but require the knowledge of Green's functions which are extremely difficult to obtain. Chen (1968, 1970) investigated the stress distribution caused by a transversely isotropic spheroidal inclusion in a transversely isotropic elastic matrix. Instead of using the Green's function formulation, Chen derived his solution using harmonic potentials, following the earlier work of Elliot (1948). In Chen (1968), an axisymmetric problem is considered, with uniform tension applied at infinity, whereas in Chen (1970) shearing loads are applied at infinity.

Although the same methodologies can be applied to study the problem of piezo-electric inclusions in an infinite piezo-electric matrix (Deeg, 1980), very few explicit solutions are available in the literature as the Green's function for the piezo-electric material cannot be obtained in closed form. Wang (1992a) used the Green's function method to show that stresses and electric induction fields are uniform inside a piezo-electric inclusion in an infinite piezo-electric matrix if spatially uniform loads are applied in the far field. He then developed a method to determine the mechanical and electric fields inside and on the surface of the inclusion. Using this method, he obtained explicit closed-form solutions for the special case of an infinitely long cylindrical inclusion. Closed form expression for the fields *outside* the spheroidal inclusion, however, *cannot* be obtained using his method as it requires the evaluation of integrals which cannot be integrated analytically.

† To whom correspondence should be addressed.

The method presented in this work allowed us to obtain closed-form solutions both inside and outside the inclusion. Our method generalizes the method used by Chen (1968). The notations used in this work follow that of Chen whenever possible. Since the problem is linear and any far-field loading can be written as a linear combination of axisymmetric loading and antisymmetric loading which is a linear combination of out-of-plane shear and in-plane shear, we can consider each of these loadings independently.

A limiting case of our solution is that of a penny-shaped crack in a piezo-electric material. A penny-shaped crack can be considered as a special case of a spheroidal void when the aspect ratio goes to infinity (Goodier, 1933; Mura, 1974). Since brittle failure of piezo-electric materials is a major consideration in the development of smart structures, it is important to characterize the fracture toughness of these materials. A necessary step in the determination of fracture toughness in brittle solids is the computation of the stress intensity factors. In this paper, explicit closed-form expressions are obtained for the stress intensity factors of a penny-shaped crack in a piezo-electric material under various remote loading conditions.

There is extensive literature on the computation of stress intensity factors in linear anisotropic elastic media [see, e.g. Shield (1951), Kassir (1968), Sih (1968), Willis (1968)]. Explicit expressions for stress intensity factors for three-dimensional problems, however, are difficult to obtain. Explicit expressions for the stress intensity factors of a flat elliptical crack in a transversely isotropic material intensity factors are given by Hoenig (1978).

The calculation of the stress intensity factors for cracks in piezo-electric materials are much more difficult as the mechanical and electric fields are coupled. Specifically, the equation of electrostatics must be considered inside the inclusion as the aspect ratio of the spheroidal hole vanishes. The formulation of crack problem in piezoelectric bodies was discussed by Pak (1990), Suo (1993). Pak considered a two-dimensional problem of a finite crack in an infinite piezo-electric medium subjected to anti-plane loading. He pointed out that the problem requires the simultaneous solution of governing equations inside and outside the crack together with continuity conditions on the crack faces. To simplify the solution procedure, Pak used an approximation which replaces the continuity of the normal component of electric induction and tangential components of electric field on the crack faces (this latter condition is equivalent to the continuity of electric potential) by a single condition of vanishing normal component of electric induction on the crack faces. Using this approximation, the solution can be obtained by just solving the field equations outside the crack. The same approximation was used by Wang (1992b) for the case of a penny-shaped crack subjected to general homogeneous far-field loading. Using the exact solution obtained in this work, we will show that Pak's approximation can lead to non-physical crack tip singularities when the electric induction field is applied at infinity.

2. AXISYMMETRIC LOADING

Consider a piezo-electric matrix containing a spheroidal piezo-electric inclusion. The geometry and loading are shown schematically on Fig. 1. The center of the inclusion is located at the origin of a cylindrical coordinate system. The z axis is chosen to coincide with the axis of symmetry of the inclusion. Both the inclusion and the matrix are assumed to be transversely isotropic. The equation of the surface of the inclusion is

$$\frac{z^2}{a^2}+\frac{r^2}{b^2}=1. \tag{1}$$

Equation (1) can be rewritten as

$$\frac{z^2}{q^2}+\frac{r^2}{q^2-1}=c^2, \tag{2}$$

where ($c^2 = a^2-b^2$) and $q = a/c$.

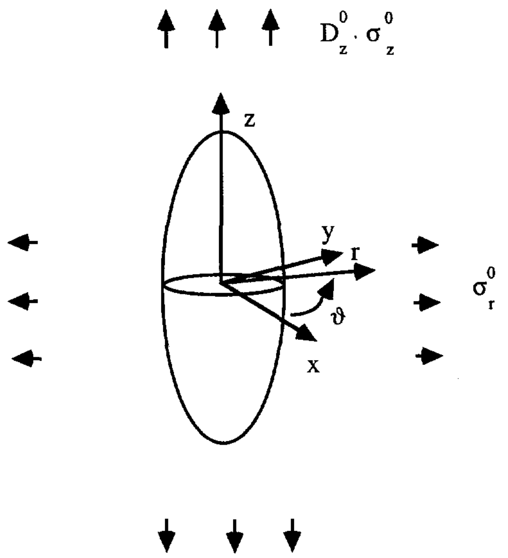

Fig. 1. A spheroidal piezo-electric inclusion in an infinite piezo-electric matrix is subjected to uniform axisymmetric far-field loading. The center of the inclusion is located at the origin of a cylindrical coordinate system. The z axis is chosen to coincide with the axis of symmetry of the inclusion. Both the inclusion and the matrix are assumed to be transversely isotropic. Constant tractions σ_r^0, σ_z^0 and electric induction component D_z^0 are imposed at infinity.

The problem is to determine the stress and electric fields inside and outside the particle when constant traction σ_r^0, σ_z^0 and electric induction component D_z^0 are imposed at infinity. The traction and electric induction give rise to the displacements $u_r^0 = \alpha_0 r, u_z^0 = \beta_0 z$ and electric potential $\varphi^0 = -\gamma_0 z$ at infinity. The constants α_0, β_0 and γ_0 are related to σ_r^0, σ_z^0 and D_z^0 through the constitutive model, i.e.

$$\begin{cases} \sigma_r^0 = (C_{11}+C_{12})\alpha_0 + C_{13}\beta_0 - e_{31}\gamma_0 \\ \sigma_z^0 = 2C_{13}\alpha_0 + C_{33}\beta_0 - e_{33}\gamma_0 \\ D_z^0 = 2e_{13}\alpha_0 + e_{33}\beta_0 + \chi_{33}\gamma_0 \end{cases} \tag{3}$$

where C_{ij} are the elastic constants of the piezo-electric matrix, e_{ij} are the piezo-electric constants, and χ_{ij} is the electric permeability.

Throughout this work, material constants and field variables associated with the inclusion will be denoted as "prime" quantities. For example, the elastic constants of the inclusion are denoted by C'_{ij}.

Axisymmetry implies that the stress and the induction field are independent of ϑ so that the equilibrium equations reduce to

$$\begin{cases} \dfrac{\partial \sigma_r}{\partial r} + \dfrac{\partial \sigma_{rz}}{\partial z} + \dfrac{\sigma_r - \sigma_\vartheta}{r} = 0 \\ \dfrac{\partial \sigma_{rz}}{\partial r} + \dfrac{\partial \sigma_z}{\partial z} + \dfrac{\sigma_{rz}}{r} = 0 \end{cases} \tag{4}$$

and the equation of electrostatics becomes

$$\frac{1}{r}\frac{\partial}{\partial r}(rD_r) + \frac{\partial D_z}{\partial z} = 0 \tag{5}$$

respectively.

The non-trivial components of the stress tensor σ_r, σ_ϑ, σ_z, σ_{rz} and the electric induction vector D_r, D_z are related to the components of the strain tensor ε_r, ε_ϑ, ε_z, ε_{rz} and to that of the electric field E_r, E_z by the following constitutive relations:

$$\begin{cases} \sigma_r = C_{11}\varepsilon_r + C_{12}\varepsilon_\vartheta + C_{13}\varepsilon_z - e_{31}E_z \\ \sigma_\vartheta = C_{12}\varepsilon_r + C_{11}\varepsilon_\vartheta + C_{13}\varepsilon_z - e_{31}E_z \\ \sigma_z = C_{13}(\varepsilon_r + \varepsilon_\vartheta) + C_{33}\varepsilon_z - e_{33}E_z \\ \sigma_{rz} = 2C_{44}\varepsilon_{rz} - e_{15}E_r \\ D_r = 2e_{15}\varepsilon_{rz} + \chi_{11}E_r \\ D_z = e_{31}(\varepsilon_r + \varepsilon_\vartheta) + e_{33}\varepsilon_z + \chi_{33}E_z. \end{cases} \tag{6}$$

The strains are related to the displacements by:

$$\varepsilon_r = \frac{\partial u_r}{\partial r}, \quad \varepsilon_\vartheta = \frac{u_r}{r}, \quad \varepsilon_z = \frac{\partial u_z}{\partial z}, \quad \varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right). \tag{7}$$

E_r, E_z are related to the electric potential φ by

$$E_r = -\frac{\partial \varphi}{\partial r}, \quad E_z = -\frac{\partial \varphi}{\partial z}. \tag{8}$$

Exactly the same equations apply for the inclusion (with C_{ij} replaced by C'_{ij} etc.).

We seek solutions of eqns (4)–(8) in the form of:

$$u_r = \frac{\partial \Phi}{\partial r}, \quad u_z = k_2\frac{\partial \Phi}{\partial z}, \quad \varphi = -k_1\frac{\partial \Phi}{\partial z} \tag{9}$$

where k_2, k_1 are unknown constants to be determined (k_2—non-dimensional, k_1—dimensional). Using eqns (6)–(9), the stresses and the induction field can be expressed in terms of Φ. Direct substitution show that eqns (4) and (5) will be satisfied if

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + v^2\frac{\partial^2 \Phi}{\partial z^2} = 0 \tag{10}$$

where

$$\begin{aligned} v^2 &= \frac{C_{44} + (C_{13}+C_{44})k_2 - (e_{31}+e_{15})k_1}{C_{11}} \\ &= \frac{C_{33}k_2 - e_{33}k_1}{C_{44}k_2 + C_{13} + C_{44} - e_{15}k_1} \\ &= \frac{e_{33}k_2 + \chi_{33}k_1}{e_{15}k_2 + e_{15} + e_{31} + \chi_{11}k_1}. \end{aligned} \tag{11}$$

Thus, (k_1, k_2) cannot be arbitrary since they must satisfy eqn (11). In the Appendix, we proved that there are only three pairs of $(k_1^{(i)}, k_2^{(i)})$ $(i = 1, 2, 3)$ that satisfy eqn (11). Thus, each pair of $(k_1^{(i)}, k_2^{(i)})$ corresponds to a v_i^2 defined by eqn (11), and a function Φ_i, which is required to satisfy the equation:

$$\frac{\partial^2 \Phi_i}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi_i}{\partial r} + v_i^2\frac{\partial^2 \Phi_i}{\partial z^2} = 0. \tag{12}$$

The functions Φ_i are harmonic functions of the variables r and $z_i = \dfrac{z}{v_i}$,

$$\frac{\partial^2 \Phi_i}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi_i}{\partial r} + \frac{\partial^2 \Phi_i}{\partial z_i^2} = 0. \tag{13}$$

The field variables can be expressed in terms of Φ_i, i.e.

$$u_r = \sum_{i=1}^{3} \frac{\partial \Phi_i}{\partial r}, \quad u_z = \sum_{i=1}^{3} \frac{k_2^{(i)}}{v_i}\frac{\partial \Phi_i}{\partial z_i}, \quad \varphi = -\sum_{i=1}^{3} \frac{k_1^{(i)}}{v_i}\frac{\partial \Phi_i}{\partial z_i}$$

$$\sigma_r = \sum_{i=1}^{3}\left(C_{11}\frac{\partial^2 \Phi_i}{\partial r^2} + C_{12}\frac{1}{r}\frac{\partial \Phi_i}{\partial r} + \frac{C_{13}k_2^{(i)} - e_{31}k_1^{(i)}}{v_i^2}\frac{\partial^2 \Phi_i}{\partial z_i^2}\right)$$

$$\sigma_\vartheta = \sum_{i=1}^{3}\left(C_{12}\frac{\partial^2 \Phi_i}{\partial r^2} + C_{11}\frac{1}{r}\frac{\partial \Phi_i}{\partial r} + \frac{C_{13}k_2^{(i)} - e_{31}k_1^{(i)}}{v_i^2}\frac{\partial^2 \Phi_i}{\partial z_i^2}\right)$$

$$\sigma_z = \sum_{i=1}^{3}\left[C_{13}\left(\frac{\partial^2 \Phi_i}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi_i}{\partial r}\right) + \frac{C_{33}k_2^{(i)} - e_{33}k_1^{(i)}}{v_i^2}\frac{\partial^2 \Phi_i}{\partial z_i^2}\right]$$

$$\sigma_{rz} = \sum_{i=1}^{3} \frac{C_{44}(1+k_2^{(i)}) - e_{15}k_1^{(i)}}{v_i}\frac{\partial^2 \Phi_i}{\partial r\, \partial z_i}$$

$$D_r = \sum_{i=1}^{3} \frac{e_{15}(1+k_2^{(i)}) + \chi_{11}k_1^{(i)}}{v_i}\frac{\partial^2 \Phi_i}{\partial r\, \partial z_i}$$

$$D_z = \sum_{i=1}^{3}\left[e_{31}\left(\frac{\partial^2 \Phi_i}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi_i}{\partial r}\right) + \frac{e_{33}k_2^{(i)} + \chi_{33}k_1^{(i)}}{v_i^2}\frac{\partial^2 \Phi_i}{\partial z_i^2}\right]. \tag{14}$$

σ_r, σ_ϑ, σ_z and D_z can be simplified using eqns (11) and (13), i.e.

$$\sigma_r = \sum_{i=1}^{3}\left[(C_{12} - C_{11})\frac{1}{r}\frac{\partial \Phi_i}{\partial r} + \frac{C_{44}(1+k_2^{(i)}) - e_{15}k_1^{(i)}}{v_i}\left(\frac{\partial^2 \Phi_i}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi_i}{\partial r}\right)\right]$$

$$\sigma_\vartheta = \sum_{i=1}^{3}\left[(C_{12} - C_{11})\frac{\partial^2 \Phi_i}{\partial r^2} + \frac{C_{44}(1+k_2^{(i)}) - e_{15}k_1^{(i)}}{v_i}\left(\frac{\partial^2 \Phi_i}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi_i}{\partial r}\right)\right]$$

$$\sigma_z = C_{44}\sum_{i=1}^{3}\left(1 + k_2^{(i)} - \frac{e_{15}}{C_{44}}k_1^{(i)}\right)\frac{\partial^2 \Phi_i}{\partial z_i^2}$$

$$D_z = e_{15}\sum_{i=1}^{3}\left(1 + k_2^{(i)} + \frac{\chi_{11}}{e_{15}}k_1^{(i)}\right)\frac{\partial^2 \Phi_i}{\partial z_i^2}. \tag{15}$$

2.1. *Solution fields inside the inclusion*

We seek a spatially uniform stress and induction field inside the inclusion. This implies that

$$u_r' = B_1 r, u_z' = B_2 z, \varphi' = B_3 z \tag{16}$$

where the B_i are constants to be determined. The stresses and induction due to these displacements and potential can be calculated using eqns (6)–(8) with C_{ij} replaced by C_{ij}'.

2.2. *Solution fields outside the inclusion*

Guided by results in potential theory, we seek harmonic functions Φ_i of the form $\Phi_i(r, z_i) = A_i H(r, z_i)$, where the A_i ($i = 1,2,3$) are constants to be determined from continuity conditions on the matrix/inclusion interface. The functions $H(r, z_i)$ are defined by:

$$H(r,z_i) = \tfrac{1}{2}[z_i^2\psi_1(q_i)+r^2\psi_2(q_i)-C_i^2\psi_0(q_i)]$$

$$\psi_0(q_i) = \tfrac{1}{2}\ln\left(\frac{q_i+1}{q_i-1}\right)$$

$$\psi_1(q_i) = \tfrac{1}{2}\ln\left(\frac{q_i+1}{q_i-1}\right)-\frac{1}{q_i}$$

$$\psi_2(q_i) = -\tfrac{1}{4}\ln\left(\frac{q_i+1}{q_i-1}\right)+\frac{1}{2}\frac{q_i}{q_i^2-1} \tag{17}$$

where the independent variable $q_i(r,z_i)$ in eqn (17) is defined implicitly by the equation

$$\frac{z_i^2}{q_i^2}+\frac{r^2}{q_i^2-1} = C_i^2 \tag{18}$$

where $C_i^2 = (a^2/v_i^2)-b^2$. Note that the functions $H(r,z_i)$ defined by eqn (17) are harmonic outside the spheroid, i.e. they satisfy eqn (13).

2.3. *Determination of the six unknown constants* B_i *and* A_i

The six unknowns B_i and A_i are determined by six equations which enforce continuity of displacement, electric potential, traction and induction across the interface. These six constants completely determine all field quantities inside and outside the inclusion. For example, the stresses and induction field inside are determined by the B_is using eqns (6)–(8) with C_{ij} replaced by C'_{ij}. Likewise, the stresses and induction outside the inclusion are determined by the A_is using eqns (6)–(9).

Let $p_i^2 = a^2/(a^2-v_i^2b^2)$. Note that eqn (18) is identical to eqn (1) when $q_i = p_i$ $(i = 1,2,3)$. In other words, $q_i = p_i$ is the condition for a point (r,z) lying on the surface of the spheroid. We will need the following identities which can be derived using eqn (18):

$$\frac{\partial q_i}{\partial r} = \frac{r}{D_i(q_i^2-1)},\quad \frac{\partial q_i}{\partial z_i} = \frac{z_i}{D_iq_i^2}, \tag{19a}$$

where

$$D_i = q_i\left(\frac{z_i^2}{q_i^4}+\frac{r^2}{(q_i^2-1)^2}\right).$$

Let n_r and n_z be the components of the unit vector normal to the surface of spheroid inclusion along the radial and axial directions, respectively. Using eqn (1),

$$n_r = \frac{r}{b^2D},\quad n_z = \frac{z}{a^2D},$$

where

$$D = \sqrt{\left(\frac{z^2}{a^4}+\frac{r^2}{b^4}\right)}.$$

Note that

$$\frac{n_z}{n_r} = \frac{b^2z}{a^2r} = \frac{q^2-1}{q^2}\,\frac{z}{r}.$$

Also, since

$$\frac{(p_i^2-1)}{p_i^2} = \frac{v_i^2 b^2}{a^2},$$

$$\frac{n_z}{n_r} = \frac{z_i}{v_i r}\frac{(p_i^2-1)}{p_i^2}. \tag{19b}$$

Continuity of displacement, electric potential, traction and induction across the interface implies that

$$[u_r] = 0, [u_z] = 0, [\varphi] = 0, [\sigma_r]n_r + [\sigma_{rz}]n_z = 0, [\sigma_{rz}]n_r + [\sigma_z]n_z = 0, [D_r]n_r + [D_z]n_z = 0$$

where $[w]$ denotes the jump of w across the interface. These continuity conditions, together with eqn (19), imply that

$$\alpha_0 + \sum_{i=1}^{3} A_i \psi_2(p_i) = B_1$$

$$\beta_0 + \sum_{i=1}^{3} \frac{k_2^{(i)}}{v_i^2} A_i \psi_1(p_i) = B_2$$

$$\gamma_0 + \sum_{i=1}^{3} \frac{k_1^{(i)}}{v_i^2} A_i \psi_1(p_i) = B_3$$

$$\sigma_r^0 + \sum_{i=1}^{3}\left[(C_{12}-C_{11})\frac{1}{r}\frac{\partial \Phi_i}{\partial r} + \frac{C_{44}(1+k_2^{(i)}) - e_{15}k_1^{(i)}}{v_i^2}\left(\frac{\partial^2 \Phi_i}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi_i}{\partial r}\right)\right]$$

$$-(C'_{11}+C'_{12})B_1 - C'_{13}B_2 + e'_{31}B_3 + \frac{n_z}{n_r}\sum_{i=1}^{3}[C_{44}(1+k_2^{(i)}) - e_{15}k_1^{(i)}]\frac{1}{v_i}\frac{\partial^2 \Phi_i}{\partial r\,\partial z_i} = 0$$

$$\sigma_z^0 + \sum_{i=1}^{3}[C_{44}(1+k_2^{(i)}) - e_{15}k_1^{(i)}]\frac{\partial^2 \Phi_i}{\partial z_i^2} - 2C'_{13}B_1 - C'_{33}B_2 + e'_{33}B_3$$

$$+\frac{n_r}{n_z}\sum_{i=1}^{3}[C_{44}(1+k_2^{(i)}) - e_{15}k_1^{(i)}]\frac{1}{v_i}\frac{\partial^2 \Phi_i}{\partial r\,\partial z_i} = 0$$

$$D_z^0 + \sum_{i=1}^{3}[e_{15}(1+k_2^{(i)}) + \chi_{11}k_1^{(i)}]\frac{\partial^2 \Phi_i}{\partial z_i^2} - 2e'_{31}B_1 - e'_{33}B_2 - \chi'_{33}B_3$$

$$+\frac{n_r}{n_z}\sum_{i=1}^{3}[e_{15}(1+k_2^{(i)}) + \chi_{11}k_1^{(i)}]\frac{1}{v_i}\frac{\partial^2 \Phi_i}{\partial r\,\partial z_i} = 0 \tag{20a}$$

In eqn (20a), all partial derivatives are evaluated on the surface of the inclusion, i.e. $q_i = p_i$. The six unknowns A_i and B_i are determined by solving the systems of six linear equations given by eqn (20a).

The following identities for points on the surface of the inclusion can be used to simplify the last three equations in eqn (20a). These identities are obtained from eqns (17) and (19), they are:

$$\frac{\partial^2 H_i}{\partial r^2} + \frac{1}{r}\frac{\partial H_i}{\partial r} + \frac{z_i}{r}\frac{p_i^2-1}{p_i^2}\frac{\partial^2 H_i}{\partial r\,\partial z_i} = -\psi_1(p_i)$$

$$\frac{\partial^2 H_i}{\partial z_i^2} + \frac{r}{z_i}\frac{p_i^2}{p_i^2-1}\frac{\partial^2 H_i}{\partial r\,\partial z_i} = -2\psi_2(p_i)$$

so that the last three equations in eqn (20a) may be rewritten as

$$\sum_{i=1}^{3} A_i\left[(C_{12}-C_{11})\psi_2(p_i)-\frac{C_{44}(1+k_2^{(i)})-e_{15}k_1^{(i)}}{v_i^2}\psi_1(p_i)\right]$$

$$= -\sigma_r^0+(C'_{11}+C'_{12})B_1+C'_{13}B_2-e'_{31}B_3$$

$$2\sum_{i=1}^{3}[C_{44}(1+k_2^{(i)})-e_{15}k_1^{(i)}]A_i\psi_2(p_i) = \sigma_z^0-2C'_{13}B_1-C'_{33}B_2+e'_{33}B_3$$

$$2\sum_{i=1}^{3}[e_{15}(1+k_2^{(i)})+\chi_{11}k_1^{(i)}]A_i\psi_2(p_i) = D_z^0-2e'_{31}B_1-e'_{33}B_2-\chi'_{33}B_3. \tag{20b}$$

Equations (20b), together with the first three equations in (20a), form a set of linear algebraic equations which allow us to determine the stress and induction everywhere.

We solve these equations for the special case of a spheroidal hole (i.e. the elastic constants of the inclusion are identically zero). The interface conditions are $[\varphi] = 0$, $\sigma_r n_r+\sigma_{rz}n_z = 0$, $\sigma_{zr}n_r+\sigma_z n_z = 0$, $[D_r]n_r+[D_z]n_z = 0$ so that eqns (20a,b) reduce to

$$\gamma_0+\sum_{i=1}^{3}\frac{k_1^{(i)}}{v_i^2}A_i\psi_1(p_i) = B_3$$

$$\sum_{i=1}^{3} A_i\left[(C_{12}-C_{11})\psi_2(p_i)-\frac{C_{44}(1+k_2^{(i)})-e_{15}k_1^{(i)}}{v_i^2}\psi_1(p_i)\right] = -\sigma_r^0$$

$$2\sum_{i=1}^{3}[C_{44}(1+k_2^{(i)})-e_{15}k_1^{(i)}]A_i\psi_2(p_i) = \sigma_z^0$$

$$2\sum_{i=1}^{3}[e_{15}(1+k_2^{(i)})+\chi_{11}k_1^{(i)}]A_i\psi_2(p_i) = D_z^0-\chi_0 B_3 \tag{21}$$

where χ_0 is the electric permeability of free space.

For the case of the spherical void in the piezo-ceramics PZT-4 subjected to the application of a transverse far-field load σ_r^0, the maximum stress concentration $\sigma_r/\sigma_r^0 = 2.75$ and is reached on the pole $z = \pm a$. If σ_z^0 is applied at infinity, the maximum stress concentration $\sigma_z/\sigma_z^0 = 1.88$ and is reached at the equator $z = 0$. When D_z^0 is imposed, the maximum induction concentration $D_z/D_z^0 = 1.51$ and is reached at the equator.

Void like defects are introduced into piezo-ceramics to improve their performance in devices such as electromechanical transducers, where high piezo-electric coefficients are desirable. A discussion of this phenomenon can be found in Dunn and Taya (1993). Stress concentration near voids in a porous piezo-ceramics can lead to premature failure of the material. Therefore, knowledge of the local stress concentration in such materials is essential for failure analysis. The stress concentration factors for different values of the aspect ratio b/a of a spheroidal void are shown in Figs 2(a,b) for piezoceramics PZT-5H. These figures provide an estimate for the local stress concentration in a porous piezo-ceramics. The material properties of PZT-4 and PZT-5H used in this work can be found in Dunn and Taya (1993) and Deeg (1980), respectively.

2.4. *Penny-shaped crack*

The solution for the case of an infinite piezo-electric body with a penny-shaped crack under homogeneous far-field loading may be obtained as a limiting case of the problem of spheroidal hole with the aspect ratio b/a approaching infinity. Expressions for the fields outside the crack, i.e. the functions Φ_i can be obtained by setting $a \to 0$ in eqn (17). In contrast with elastic crack problems, the equation of electrostatics must be considered inside the inclusion as $a \to 0$. Formulation of the problem of piezo-electric body with crack was discussed by Pak (1990). He considered a two-dimensional problem of a finite crack in an infinite piezo-electric medium subjected to antiplane loading. Pak pointed out that the

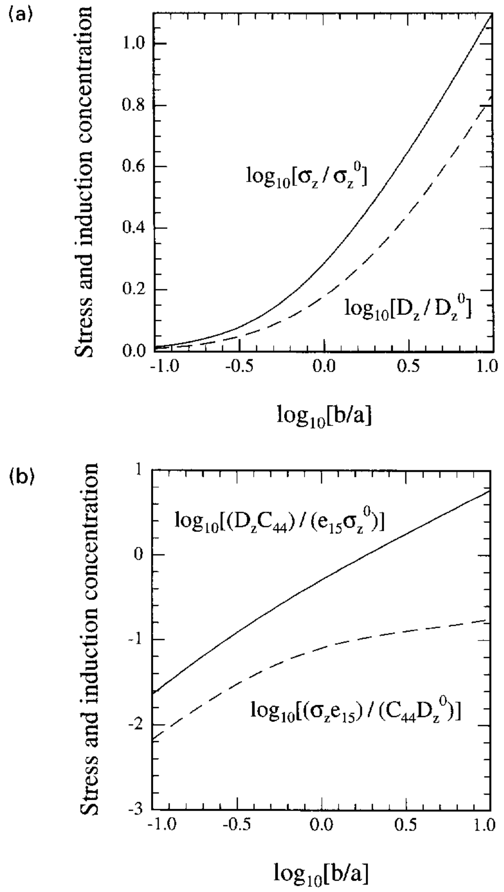

Fig. 2. A spheroidal void $(z^2/a^2)+(r^2/b^2) = 1$ is subjected to uniform axisymmetric far-field loading. The stress concentration factors for different values of the aspect ratio b/a of a spheroidal void are shown for the case when matrix is made of the piezo-ceramics PZT-5H. To calculate the stress concentration factors, only σ_z^0 or D_z^0 are applied at infinity. All other components of the far-field loading are set equal to zero. (a) σ_z/σ_z^0 is plotted as a solid line and D_z/D_z^0 is plotted as a dashed line. (b) $(D_z/e_{15})(C_{44}/\sigma_z^0)$ is plotted as a solid line and $(\sigma_z/C_{44})(e_{15}/D_z^0)$ is plotted as a dashed line.

solution requires simultaneous solution of governing equations inside and outside the crack together with continuity conditions on the crack faces. However, to simplify the solution procedure, Pak used an approximation which replaces the continuity of the normal component of electric induction and tangential components of electric field on the crack faces (this latter condition is equivalent to continuity of potential) by a single condition of vanishing normal component of electric induction on the crack faces. Using this approximation, the solution was obtained by solving the field equations outside the crack. The same approximation was used by Wang (1992a) for the case of a penny-shaped crack. This approximation gives rise to a non-physical singularities in electric induction near the crack edge.

For example, consider the special case of no far-field mechanical loading (i.e. traction free boundary condition imposed at infinity) with the electric induction D_z^0 applied at infinity. The solution in this case is given by the homogeneous field of induction D_z^0 and zero stresses everywhere, i.e. the A_i are all zero. It is easy to verify that this solution satisfies all the governing equations and continuity conditions. Physically, this means that induction loading with zero traction boundary conditions in the far-field cannot contribute to singular stresses and induction at the crack edge. Note that the induction gives rise to stress free residual strains ε_{ij}^R in the piezo-electric material. ε_{ij}^R can be obtained by solving eqn (6) with $\sigma_{ij} = 0$, $D_r = 0$ and $D_z = D_z^0$. This example illustrates the need to use the exact continuity conditions in the solution of crack problems. It should be noted that this example does not imply that there is no singular field at the crack tip if homogeneous displacement boundary conditions are imposed at infinity, as we shall comment below.

Thus, the crack problem is still coupled as in the case of an inclusion. Specifically, traction continuity implies that $\sigma_z = 0$, $\sigma_{rz} = 0$ on the crack faces. In addition, the normal component of electric induction D_z and the tangential component of electric field E_r are continuous across the crack faces. These conditions will be satisfied if constants A_i satisfy the following system of linear equations:

$$\frac{\pi i}{2}\sum_{i=1}^{3} A_i[C_{44}(1+k_2^{(i)})-e_{15}k_1^{(i)}] = \sigma_z^0$$

$$\sum_{i=1}^{3} A_i\frac{[C_{44}(1+k_2^{(i)})-e_{15}k_1^{(i)}]}{v_i} = 0$$

$$\frac{\pi i}{2}\sum_{i=1}^{3} A_i[e_{15}(1+k_2^{(i)})+\chi_{11}k_1^{(i)}] = D_{z^0}-\chi_0 B_3$$

$$\sum_{i=1}^{3}\frac{k_1^{(i)}}{v_i}A_i = 0 \tag{22}$$

where B_3 determines electric field inside the crack as given by eqn (16). Using eqns (15) and (21), it can be shown that the stress distribution in the crack plane ($z = 0$) is:

$$\sigma_z(r,0) = \sigma_z^0\left[\frac{2}{\pi\sqrt{\left(\frac{r}{b}\right)^2-1}}+\frac{2}{\pi}\tan^{-1}\sqrt{\left(\frac{r}{b}\right)^2-1}\right],\quad r > b. \tag{23}$$

The z component of electric induction is found to be

$$D_z(r,0) = (D_z^0-\chi_0 B_3)\left[\frac{2}{\pi\sqrt{\left(\frac{r}{b}\right)^2-1}}+\frac{2}{\pi}\tan^{-1}\sqrt{\left(\frac{r}{b}\right)^2-1}\right]+\chi_0 B_3,\quad r > b. \tag{24}$$

If we define the Mode I intensity factors K_I^σ and K_I^D by

$$\sigma_z\sqrt{2\pi(r-b)}\xrightarrow[r\to b]{} K_I^\sigma$$

$$D_z\sqrt{2\pi(r-b)}\xrightarrow[r\to b]{} K_I^D$$

then they are related to the far-field loading by

$$K_I^\sigma = \frac{2\sigma_z^0\sqrt{b}}{\sqrt{\pi}},\quad K_I^D = \frac{2(D_z^0-\chi_0 B_3)\sqrt{b}}{\sqrt{\pi}}. \tag{25a}$$

When $\sigma_z^0 = 0$, it is easy to see that $D_z^0-\chi_0 B_3 = 0$, so that $K_I^\sigma = K_I^D = 0$. In other words, the intensity factors are determined solely by the far-field traction. It should be noted that eqn (25a) agrees with the Mode I stress intensity factor obtained by Hoenig (1978) for the special case of a transversely isotropic elastic material. Note that K_I^σ is independent on material properties whereas K_I^D is dependent on material properties through χ_0 and B_3.

Some important loading cases may be reduced to the one considered here, for example, when the surface of the body is clamped (i.e. zero displacements at infinity) and an induction

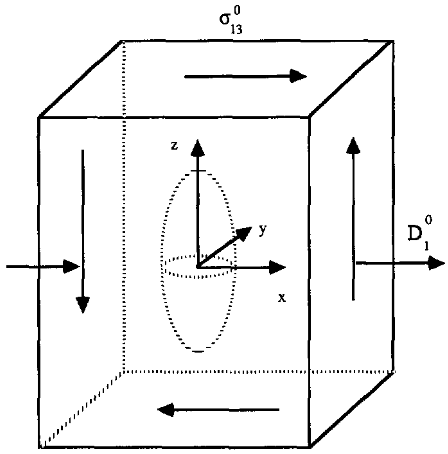

Fig. 3. A spheroidal piezo-electric inclusion in an infinite piezo-electric matrix is subjected to uniform antisymmetric far-field loading. The case under consideration is out-of-plane shear. Constant traction σ_{13}^0 and induction D_1^0 are imposed at infinity.

field D_z^0 is applied at infinity. If we denote corresponding intensity factors as $\tilde{K}_{\mathrm{I}}^{\sigma}$ and $\tilde{K}_{\mathrm{I}}^{D}$, then

$$\tilde{K}_{\mathrm{I}}^{\sigma} = -\frac{e_{33}}{\chi_{33}}\frac{D_z^0}{\sigma_z^0}K_{\mathrm{I}}^{\sigma}, \quad \tilde{K}_{\mathrm{I}}^{D} = -\frac{e_{33}}{\chi_{33}}\frac{D_z^0}{\sigma_z^0}K_{\mathrm{I}}^{D}. \tag{25b}$$

As we can see, the singularities in stress and electric induction are proportional to applied electric loading and are dependent on material properties.

3. ANTISYMMETRIC CASE: IN-PLANE AND ANTI-PLANE SHEAR

In this case shearing tractions σ_{13}^0 and σ_{12}^0 and the induction D_1^0 are imposed at infinity. The geometry and loading are shown schematically on Fig. 3. The full set of equilibrium equations is used:

$$\begin{cases} \dfrac{\partial \sigma_r}{\partial r} + \dfrac{1}{r}\dfrac{\partial \sigma_{r\vartheta}}{\partial \vartheta} + \dfrac{\partial \sigma_{rz}}{\partial z} + \dfrac{\sigma_r - \sigma_\vartheta}{r} = 0 \\ \dfrac{\partial \sigma_{r\vartheta}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \sigma_\vartheta}{\partial \vartheta} + \dfrac{\partial \sigma_{\vartheta z}}{\partial z} + \dfrac{2}{r}\sigma_{r\vartheta} = 0 \\ \dfrac{\partial \sigma_{rz}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \sigma_{\vartheta z}}{\partial \vartheta} + \dfrac{\partial \sigma_z}{\partial z} + \dfrac{\sigma_{rz}}{r} = 0. \end{cases}$$

The equation of electrostatics is

$$\frac{\partial D_r}{\partial r} + \frac{1}{r}\frac{\partial D_\vartheta}{\partial \vartheta} + \frac{\partial D_z}{\partial z} + \frac{D_r}{r} = 0.$$

Additional constitutive relations must be appended to those given in eqn (6), i.e.

$$\begin{aligned} \sigma_{z\vartheta} &= 2C_{44}\varepsilon_{z\vartheta} - e_{15}E_\vartheta \\ \sigma_{r\vartheta} &= (C_{11} - C_{12})\varepsilon_{r\vartheta} \\ D_\vartheta &= 2e_{15}\varepsilon_{\vartheta z} + \chi_{11}E_\vartheta. \end{aligned} \tag{26}$$

The strains ε_r, ε_z, ε_{rz} are related to displacements by eqn (7) and the strain components ε_ϑ, $\varepsilon_{r\vartheta}$, $\varepsilon_{z\vartheta}$ by

$$\varepsilon_\vartheta = \frac{1}{r}\frac{\partial u_\vartheta}{\partial \vartheta} + \frac{u_r}{r}, \quad \varepsilon_{z\vartheta} = \frac{1}{2}\left(\frac{\partial u_\vartheta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \vartheta}\right), \quad \varepsilon_{r\vartheta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \vartheta} + \frac{\partial u_\vartheta}{\partial r} - \frac{u_\vartheta}{r}\right). \tag{27}$$

The electric field is related to the electric potential by eqn (8) and

$$E_\vartheta = -\frac{1}{r}\frac{\partial \varphi}{\partial \vartheta}. \tag{28}$$

We seek a solution of the equilibrium equations and equations of electrostatics in the form of

$$u_r = \frac{\partial \Phi}{\partial r} + \frac{1}{r}\frac{\partial \Psi}{\partial \vartheta}, \quad u_\vartheta = \frac{1}{r}\frac{\partial \Phi}{\partial \vartheta} - \frac{\partial \Psi}{\partial r}, \quad u_z = k_2\frac{\partial \Phi}{\partial z}, \quad \varphi = -k_1\frac{\partial \Phi}{\partial z} \tag{29}$$

where (k_1, k_2) are constants. The stresses and the electric induction can be expressed in terms of Φ and Ψ using eqns (6)–(8) and (26)–(29). Substituting these expressions into the equilibrium equations, we find

$$\Delta_0\Phi + v^2\frac{\partial^2\Phi}{\partial z^2} = 0, \quad \Delta_0\Psi + v_4^2\frac{\partial^2\Psi}{\partial z^2} = 0 \tag{30}$$

where v^2 is defined by eqn (11), with

$$v_4^2 = \frac{2C_{44}}{C_{11}-C_{12}},$$

$$\Delta_0 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \vartheta^2}.$$

As in the axisymmetric case, eqn (29) is a solution if (k_1, k_2) satisfy eqn (11).

Besides the variables $z_i = z/v_i$ ($i = 1,2,3$), we introduce a new variable $z_4 = z/v_4$. In terms of these variables, Φ_i satisfy the Laplace equation

$$\Delta_0\Phi_i + \frac{\partial^2\Phi_i}{\partial z_i^2} = 0 \tag{31a}$$

and Ψ satisfies the Laplace equation

$$\Delta_0\Psi + \frac{\partial^2\Psi}{\partial z_4^2} = 0. \tag{31b}$$

3.1. *Out-of-plane shear*

Consider first the case of out of plane shear where σ_{13}^0, D_1^0 are prescribed at infinity. The corresponding non-trivial displacement and potential are $u_z^0 = \alpha_0 r\cos\vartheta$ and $\varphi^0 = \beta_0 r\cos\vartheta$, respectively. The constants α_0 and β_0 are related to σ_{13}^0 and D_1^0 by the constitutive law

$$C_{44}\alpha_0 + e_{15}\beta_0 = \sigma_{13}^0$$

$$e_{15}\alpha_0 - \chi_{11}\beta_0 = D_1^0. \tag{32}$$

3.2. *Solution fields inside the inclusion*

The assumption of spatially uniform fields inside the inclusion implies that

$$u'_r = B_1 z\cos\vartheta, \quad u'_\vartheta = -B_1 z\sin\vartheta, \quad u'_z = B_2 r\cos\vartheta, \quad \varphi' = B_3 r\cos\vartheta. \tag{33}$$

The corresponding uniform stresses and electric induction can be obtained from eqn (33) using eqns (6) and (26) with the C_{ij} and χ_{ij} replaced by C'_{ij} and χ'_{ij}.

3.3. *Solution fields outside the inclusion*

We seek Φ_i and Ψ of the form

$$\Phi_i = A_i H(r, z_i)\cos(\vartheta), \quad \Psi = A_4 H(r, z_4)\sin(\vartheta) \tag{34}$$

where

$$H(r, z_\alpha) = rz_\alpha[\phi(q_\alpha) + \psi(q_\alpha)] + \frac{2z_\alpha C_\alpha^2}{rq_\alpha}$$

$$\phi(q_\alpha) = \frac{3}{q_\alpha}\left[1 + \frac{q_\alpha}{2}\ln\left(\frac{q_\alpha - 1}{q_\alpha + 1}\right)\right], \quad \psi(q_\alpha) = \frac{1}{q_\alpha(q_\alpha^2 - 1)}$$

where the independent variables q_α are defined by

$$\frac{z_{\alpha^2}}{q_\alpha^2} + \frac{r^2}{(q_\alpha^2 - 1)} = C_\alpha^2, \quad \alpha = (1,2,3,4)$$

$$C_4^2 = \frac{a^2}{v_4^2 - b^2}.$$

The v_i are defined by eqn (11) and v_4 is defined by eqn (30). The stresses and induction can be obtained from Φ_i and Ψ using eqns (6)–(8) and (26)–(29).

3.4. *Determination of the unknown constants* B *and* A

The seven unknowns B_i and A_i are determined by seven equations which remove singularity in stresses at $r = 0$, enforce continuity of displacement, traction and induction across the interface. These seven constants completely determine all field quantities inside and outside the inclusion.

Let $p_4^2 = a^2/(a^2 - v_4^2 b^2)$. Using the fields inside and outside the inclusion derived in the previous sections, the continuity conditions $[u_r] = 0$, $[u_\vartheta] = 0$, $[u_z] = 0$, $[\varphi] = 0$, $[\sigma_r]n_r + [\sigma_{rz}]n_z = 0$, $[\sigma_{\vartheta r}]n_r + [\sigma_{\vartheta z}]n_z = 0$, $[\sigma_{rz}]n_r + [\sigma_z]n_z = 0$, $[D_r]n_r + [D_z]n_z = 0$ on the interface are:

$$\sum_{i=1}^{3} A_i C_i^3 - A_4 C_4^3 = 0$$

$$\sum_{i=1}^{3} \frac{A_i}{v_i}\phi(p_i) + \frac{A_4}{v_4}\phi(p_4) = B_1$$

$$\sum_{i=1}^{3} \frac{k_2^{(i)} A_i}{v_i}[\phi(p_i) + 3\psi(p_i)] + \alpha_0 = B_2$$

$$-\sum_{i=1}^{3}\frac{k_1^{(i)}A_i}{v_i}[\phi(p_i)+3\psi(p_i)]+\beta_0 = B_3$$

$$\sum_{i=1}^{3}\frac{1+k_2^{(i)}-\dfrac{e_{15}}{C_{44}}k_1^{(i)}}{v_i}A_i[\phi(p_i)+3\psi(p_i)]+\frac{1}{v_4}A_4(\phi(p_4)+3\psi(p_4))$$
$$+\frac{\sigma_{13}^0}{C_{44}}-\frac{C'_{44}}{C_{44}}(B_1+B_2)-\frac{e'_{15}}{C_{44}}B_3 = 0$$

$$\sum_{i=1}^{3}\frac{1+k_2^{(i)}-\dfrac{e_{15}}{C_{44}}k_1^{(i)}}{v_i}A_i\psi(p_i) = 0$$

$$\sum_{i=1}^{3}\frac{1+k_2^{(i)}+\dfrac{\chi_{11}}{e_{15}}k_1^{(i)}}{v_i}A_i[\phi(p_i)-3\psi(p_i)]+\frac{1}{v_4}A_4[\phi(p_4)+3\psi(p_4)]$$
$$+\frac{D_1^0}{e_{15}}-\frac{e'_{15}}{e_{15}}(B_1+B_2)+\frac{\chi'_{11}}{e_{15}}B_3 = 0 \quad (35)$$

Note that there are only seven independent equations since two pairs of the continuity equations are linearly dependent (the first and the second, the fifth and the sixth).

As before, we consider the special case of a spheroidal void. For this case eqns (35) become

$$\sum_{i=1}^{3}A_iC_i^3-A_4C_4^3 = 0$$

$$-\sum_{i=1}^{3}\frac{k_1^{(i)}A_i}{v_i}[\phi(p_i)+3\psi(p_i)]+\beta_0 = B_3$$

$$\sum_{i=1}^{3}\frac{1+k_2^{(i)}-\dfrac{e_{15}}{C_{44}}k_1^{(i)}}{v_i}A_i[\phi(p_i)+3\psi(p_i)]+\frac{1}{v_4}A_4[\phi(p_4)+3\psi(p_4)]+\frac{\sigma_{13}^0}{C_{44}} = 0$$

$$\sum_{i=1}^{3}\frac{1+k_2^{(i)}-\dfrac{e_{15}}{C_{44}}k_1^{(i)}}{v_i}A_i\psi(p_i) = 0$$

$$\sum_{i=1}^{3}\frac{1+k_2^{(i)}+\dfrac{\chi_{11}}{e_{15}}k_1^{(i)}}{v_i}A_i[\phi(p_i)-3\psi(p_i)]+\frac{1}{v_4}A_4[\phi(p_4)+3\psi(p_4)]+\frac{D_1^0}{e_{15}}+\frac{\chi_0}{e_{15}}B_3 = 0 \quad (36)$$

Figure 4 shows the dependence of σ_r/σ_{13}^0, $\sigma_\vartheta/\sigma_{13}^0$, σ_z/σ_{13}^0, $\sigma_{rz}/\sigma_{13}^0$ on the latitude $\omega = sin^{-1}(z/r)$. The quantities σ_r/σ_{13}^0, $\sigma_\vartheta/\sigma_{13}^0$, σ_z/σ_{13}^0, $\sigma_{rz}/\sigma_{13}^0$ are evaluated on the boundary of a spherical hole inside the piezo-ceramics PZT-4 at $\vartheta = 0$. The location of crack initiation in a porous piezo-ceramics can be determined using Fig. 4 and the appropriate failure criterion.

3.5. *Penny-shaped crack*

We will follow the same strategy as in axisymmetric case. The electric field inside the crack has no z component, so the z component of electric induction on the crack faces is zero. The electric field inside the crack is constant, because the crack is a limiting case of inclusion and may be adjusted to provide continuity of tangential components of the electric field. The solution also has to satisfy the traction continuity condition $\sigma_z = 0$, $\sigma_{rz} = 0$ on

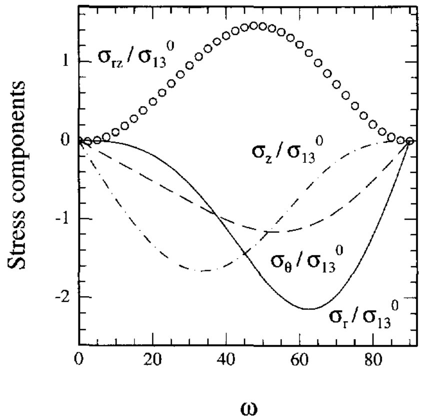

Fig. 4. A spheroidal void $(z^2/a^2)+(r^2/b^2) = 1$ is subjected to uniform antisymmetric far-field loading (out-of-plane shear). The matrix is the piezo-ceramics PZT-4. Normalized stress components σ_r/σ_{13}^0, $\sigma_\vartheta/\sigma_{13}^0$, σ_z/σ_{13}^0, $\sigma_{rz}/\sigma_{13}^0$ are plotted along the boundary of the void at $\vartheta = 0$ for different values of the latitude $\omega = \sin^{-1}(z/r)$. σ_r/σ_{13}^0 is plotted as a solid line, $\sigma_\vartheta/\sigma_{13}^0$ is plotted as a dashed line, σ_z/σ_{13}^0 is plotted as $-\cdot-\cdot-$ and $\sigma_{rz}/\sigma_{13}^0$ as ○○○○○○.

the crack faces. Using these conditions and the condition of no stress singularity at $r = 0$, we obtain a system of equations on constants A_i:

$$\sum_{i=1}^{3}\left(1+k_2^{(i)}+\frac{\chi_{11}}{e_{15}}k_1^{(i)}\right)A_i = 0$$

$$\sum_{i=1}^{3}\left(1+k_2^{(i)}-\frac{e_{15}}{C_{44}}k_1^{(i)}\right)A_i = 0$$

$$\tfrac{3}{2}C_{44}\pi i\left[\sum_{i=1}^{3}\frac{\left(1+k_2^{(i)}-\frac{e_{15}}{C_{44}}k_1^{(i)}\right)A_i}{v_i}+\frac{A_4}{v_4}\right]+\sigma_{13}^0 = 0$$

$$\sum_{i=1}^{3}A_i = A_4. \tag{37}$$

After constants A_i have been determined, stresses and electric induction everywhere outside the crack may be calculated according to eqns (6)–(8) and (26)–(29).

In particular, stress distribution at $z = 0$, $\vartheta = 0$ is given by

$$\sigma_{13} = \sigma_{13}^0\left(\frac{2}{\pi\sqrt{\left(\frac{r}{b}\right)^2-1}}+\frac{2}{\pi}\tan^{-1}\sqrt{\left(\frac{r}{b}\right)^2-1}+\frac{2b^2}{\pi r^2\sqrt{\left(\frac{r}{b}\right)^2-1}}\right)+C_{44}\frac{A_4}{v_4}\frac{6b^2 i}{r^2\sqrt{\left(\frac{r}{b}\right)^2-1}}. \tag{38}$$

If we define Mode II and Mode III stress intensity factors K_{II}^σ, K_{III}^σ as

$$\sigma_{rz}\sqrt{2\pi(r-b)}\xrightarrow[r\to b]{}K_{\mathrm{II}}^\sigma$$

$$\sigma_{z\vartheta}\sqrt{2\pi(r-b)}\xrightarrow[r\to b]{}K_{\mathrm{III}}^\sigma$$

then

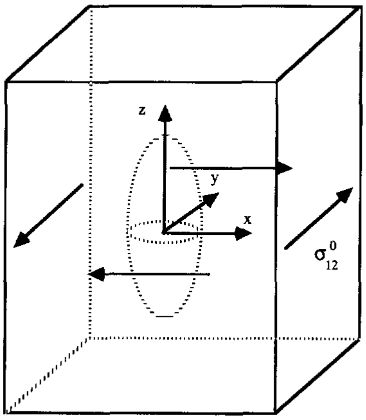

Fig. 5. A spheroidal piezo-electric inclusion in an infinite piezo-electric matrix is subjected to uniform antisymmetric far-field loading. The case under consideration is in-plane shear. The constant traction σ_{12}^0 is imposed at infinity.

$$K_{\rm II}^{\sigma} = \left(\sigma_{13}^0 \frac{4\sqrt{b}}{\sqrt{\pi}} + C_{44}\frac{A_4}{v_4}6i\sqrt{\pi b}\right)\cos(\vartheta)$$

$$K_{\rm III}^{\sigma} = C_{44}\frac{A_4}{v_4}6i\sqrt{\pi b}\sin(\vartheta) \tag{39}$$

In the case of out-of-plane shear, our calculation shows that there is no singularity in the induction field near the crack tip. Furthermore, our calculation shows that far-field electric loading can not contribute to stress singularities at the crack tip. The same result was obtained by Wang (1992b). This can also be justified by the fact that a homogeneous horizontal field of electric induction and zero stresses everywhere satisfy all the governing equations and the boundary conditions on the crack faces.

3.6. *In-plane shear*

For this case σ_{12}^0 is imposed at infinity. The geometry and loading are shown schematically in Fig. 5. The displacements inside the inclusion are $u_r' = B_1 r\sin 2\vartheta, u_\vartheta' = B_1 r\cos 2\vartheta$. The potential is constant inside the inclusion.

Outside the inclusion, we seek Φ_i and Ψ of the form

$$\Phi_i = A_i H(r, z_i)\sin 2\vartheta, \quad \Psi = A_4 H(r, z_4)\cos 2\vartheta,$$

where

$$H(r, z_\alpha) = r^2\left[\varsigma(q_\alpha) - \frac{2q_\alpha}{(q_\alpha^2-1)^2}\right] + \frac{8C_\alpha^2 q_\alpha}{r^2}\left(C_\alpha^2 + \frac{z_\alpha^2}{q_\alpha^2}\right),$$

$$\varsigma(q_\alpha) = 3\left[\frac{1}{2}\ln\left(\frac{q_\alpha-1}{q_\alpha+1}\right) + \frac{q_\alpha}{q_\alpha^2-1}\right], \quad (\alpha = 1,2,3,4). \tag{40}$$

As before, continuity of displacement, traction and induction at the interface gives

$$\sum_{i=1}^{3}\frac{k_2^{(i)}}{v_i}A_i C_i^3 = 0$$

$$\sum_{i=1}^{3}\frac{k_1^{(i)}}{v_i}A_i C_i^3 = 0$$

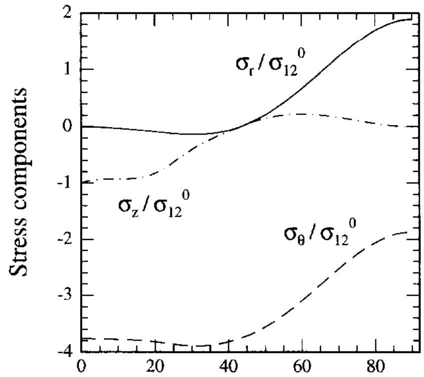

Fig. 6. A spheroidal void $(z^2/a^2)+(r^2/b^2) = 1$ is subjected to uniform antisymmetric far-field loading (in-plane shear). Matrix is made of the piezo-ceramics PZT-4. Normalized stress components σ_r/σ_{12}^0, $\sigma_\vartheta/\sigma_{12}^0$, σ_z/σ_{12}^0 are plotted along the boundary of the void at $\vartheta = \pi/4$ for different values of the latitude $\omega = \sin^{-1}(z/r)$. σ_r/σ_{12}^0 is plotted as a solid line, $\sigma_\vartheta/\sigma_{12}^0$ is plotted as a dashed line, σ_z/σ_{12}^0 is plotted as $-\cdot-\cdot-$.

$$\sum_{i=1}^{3}\frac{A_i}{v_i}C_i^3+\frac{A_4}{v_4}C_4^3=0$$

$$2\sum_{i=1}^{3}A_i\varsigma(p_i)-2A_4\varsigma(p_4)+\frac{\sigma_{12}^0}{C_{11}-C_{12}}-B_1=0$$

$$\frac{24}{v_4^2}\frac{A_4}{p_4(p_4^2-1)}+\frac{C_{11}-C'_{11}-C_{12}+C'_{12}}{C_{44}}B_1=0 \tag{41}$$

In the case of spheroidal void eqn (41) simplifies to

$$\sum_{i=1}^{3}\frac{k_2^{(i)}}{v_i}A_iC_i^3=0$$

$$\sum_{i=1}^{3}\frac{k_1^{(i)}}{v_i}A_iC_i^3=0$$

$$\sum_{i=1}^{3}\frac{A_i}{v_i}C_i^3+\frac{A_4}{v_4}C_4^3=0$$

$$2\sum_{i=1}^{3}A_i\varsigma(p_i)-A_4\left[\varsigma(p_4)-\frac{6}{p_4(p_4^2-1)}\right)\Bigg]+\frac{\sigma_{12}^0}{C_{11}-C_{12}}=0. \tag{42}$$

Figure 6 shows the dependence of σ_r/σ_{12}^0, $\sigma_\vartheta/\sigma_{12}^0$, σ_z/σ_{12}^0 on the latitude $\omega = \sin^{-1}(z/r)$. The quantities σ_r/σ_{12}^0, $\sigma_\vartheta/\sigma_{12}^0$, σ_z/σ_{12}^0 are evaluated on the boundary of a spherical hole inside the piezo-ceramics PZT-4 at $\vartheta = \pi/4$. The direction $\vartheta = \pi/4$ is chosen since this corresponds to the direction of maximum principal stress. The location of crack initiation in a porous piezo-ceramics under in-plane shear can be determined using Fig. 6 and appropriate failure criterion.

Solution of the penny-shaped crack problem for this case is trivial, because the homogeneous stress field σ_{12}^0 satisfies all necessary equations.

4. CONCLUSION

The problem of a spheroidal inclusion in an infinite piezo-electric matrix under general homogeneous loading is solved in closed form using harmonic functions. The solution involves first finding the roots of a cubic polynomial eqn (11) and then solving a set of linear algebraic equations to determine the coefficients of the harmonic functions inside or outside the inclusion. Although it is always possible to find the roots of a cubic polynomial in a closed form, in practice, it is easier to write a small program to compute these roots and to solve the linear system of algebraic equations.

Closed-form solutions are also obtained for the limiting case of a penny-shape crack. Explicit expressions for the stress intensity factors under different loading conditions are given; these are often used to characterize material toughness. Furthermore, the complete traction field is found in closed form which involves only elementary functions. Formulas for components of stress and electric induction directly ahead of the crack front are given explicitly. The stress distribution, together with a micromechanics model of failure, can also be used to predict fracture. Our analysis shows that the approximation suggested by Pak (1990), which replaces the continuity of the normal component of electric induction and tangential components of electric field on the crack faces by a single condition of vanishing normal component of electric induction on the crack faces, leads to non-physical singularities in electric induction and stresses near the crack edge. For example, we have demonstrated that for a traction free infinite body with electric induction D_z^0 applied at infinity, the stresses are zero everywhere and the induction is homogeneous, whereas the use of the approximate boundary condition will lead to a singular induction field and stresses at the crack edge. Similar result was obtained by Suo (1993) for the case of a plane conducting crack.

Such a contradiction appears because Pak assumes that since the normal component of an electric field outside the crack is very small compared to the normal component of the electric field inside the crack—equations (A10) and (A11) in Pak (1990)—the normal component of the electric field outside the crack may be set equal to zero. In this approximation he ignores the possibility of having a large electric field inside the crack and a finite field outside. This is exactly the case, as we see from the exact solution. The fact that the homogeneous electric induction field, which is a solution for the traction free case, fails to satisfy the approximate condition proposed by Pak means that his approximation is not justified and leads to non-physical results.

It should be noted that the singular induction field can exist at the crack edge under traction free boundary conditions if the electric permeability of the inclusion is exactly zero. This limit may not be important since the permeability is bounded from below by that of vacuum, which is non-zero.

We note that the solution for a traction free finite body with an arbitrary number of flat cracks subjected to a constant electric induction applied at its two ends will be stress free with uniform induction everywhere provided that the body geometry is sufficiently simple so that the applied electric induction field satisfies all the boundary conditions. For example, a rectangular block of piezo-electric material with cracks oriented parallel to one face of the block subjected to an applied induction normal to this face will satisfy this criterion. Obviously, the cracks can have any shape as long as they are planar.

The solution given in this paper can also be used to predict effective properties of a piezo-composite reinforced by spheroidal inclusions. The procedure itself was not carried out in this paper, but such methods are well known and easy to apply when a closed-form solution for a single inclusion in an infinite matrix is known.

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APPENDIX

To show that there are only three pairs of $(k_1^{(i)}, k_2^{(i)})$ $(i = 1,2,3)$ that satisfy eqn (11), we introduce new variables m_1 and m_2:

$$m_1 = C_{44}k_2 + C_{13} + C_{44} - e_{15}k_1$$

$$m_2 = C_{44} + (C_{13} + C_{44})k_2 - (e_{31} + e_{15})k_1.$$

Since m_1 and m_2 are linearly dependent on k_1 and k_2, the second equality in eqn (11) may be rewritten in the form $m_1 m_2 = L(m_1, m_2)$, where $L(m_1, m_2)$ is a linear function. This implies that $m_2 = L_1(m_1)/L_2(m_1)$, where $L_1(m_1)$, $L_2(m_1)$ are linear functions. The third equality in eqn (11) may be rewritten as

$$\frac{C_{11}}{l_1(m_1, m_2)} = \frac{m_2}{l_2(m_1, m_2)} \tag{A1}$$

where $l_1(m_1, m_2)$, $l_2(m_1, m_2)$ are linear functions. Substituting $m_2 = L_1(m_1)/L_2(m_1)$ in (A1), we obtain

$$\frac{C_{11}L_2(m_1)}{l_1^*(m_1)} = \frac{L_1(m_1)}{l_2^*(m_1)} \tag{A2}$$

where $l_1^*(m_1)$, $l_2^*(m_1)$ are quadratic functions. Equation (A2) implies that m_1 satisfies a cubic equation, i.e.

$$C_{11}L_2(m_1)l_2^*(m_1) - L_1(m_1)l_1^*(m_1) = 0.$$

Since $m_2 = L_1(m_1)/L_2(m_1)$, the system (11) has only three solutions $(k_1^{(i)}, k_2^{(i)})$, $(i = 1,2,3)$.